

WAVE STRUCTURE AND COHERENCE RADIUS OF PLANE WAVES PROPAGATING IN ATMOSPHERIC TURBULENCE

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Abstract- Based on the weak fluctuation theory, the new analytic expressions of wave structure for the plane wave propagation through atmospheric turbulence in a horizontal path are derived. Moreover, using the Rytov approximation method, the new expressions for coherence radius are obtained. It is shown that they are influenced by the power law exponent α , the anisotropic factor ζ , and the turbulence strength σ_R^2 .

Keywords –Wave structure, Coherence radius, Plane wave propagation, Atmospheric turbulence

1. INTRODUCTION

Beam spreading caused by atmospheric turbulence occupies a very important position in free space optics (FSO) communication systems, because it determines the loss of the power at the receiver plane [1]. For the optical waves propagation, the classic Kolmogorov model has been widely used in theoretical researches due to its simple mathematical form [1], [2]. Over the years, the Kolmogorov model is extended, and several non-Kolmogorov turbulence models have been also proposed [3]–[5]. Toselli is one of them and he analyses the angle of arrival fluctuations by using the generalized exponent factor α instead of the standard exponent value 11/3 [3]. The anisotropic factor is also used to describe anisotropy of the atmosphere turbulence [6], and the generalized non-Kolmogorov von Karman spectrum of the anisotropic atmospheric turbulence is available [4], [5]. In addition, there are also numerous studies on beam wanders, loss of spatial coherence, and the angle of arrival fluctuation [7]– [9], which are all related to the random fluctuation of optical waves propagating through random media.

Lately, more research attention is drawn to the theoretical survey of wave structure function (WSF) for the long-exposure modulation transfer function (MTF) and spatial coherence radius (SCR) [10]– [13]. Based on the Rytov approximation method, a researcher like Young proposes new expressions for the WSFs of optical waves, which fit to the moderate to strong fluctuation regimes [10]. Lu has derived new expressions for the WSFs and the SCRs for plane waves and spherical waves propagating through a homogeneous and isotropic oceanic turbulence [11]. Moreover, Cui considers the turbulence scales and has derived the long-exposure MTFs for plane waves and spherical waves propagating through anisotropic non-Kolmogorov atmospheric turbulence [12]; Kotiang and Choi also have derived a new long-exposure MTF for Gaussian waves propagating through isotropic non-Kolmogorov atmospheric turbulence [13].

In this study, we derive new WSF and SCR expressions for the plane waves and the spherical waves which propagate in the anisotropic non-Kolmogorov atmosphere turbulence. Here, the generalized von Karman model is used by incorporating the anisotropic factor $\zeta 2-\alpha$. In the simulation analyses, using the newly derived WSF and SCR expressions, the effects of the turbulence strength are investigated. In addition to the influences of the power law exponent, and the anisotropic factor, which are all affecting parameters of the WSFs and SCRs are also carefully analyzed.

2. ANISOTROPIC NON-KOLMOGOROV SPECTRUM WITH INNER AND OUTER SCALES

The conventional isotropic non-Kolmogorov spectrum is as follows [14]:

$$\Phi_n(\kappa, \alpha) = A(\alpha) \tilde{C}_n^2 \kappa^{-\alpha}, (\kappa > 0, 3 < \alpha < 4), \quad (1)$$

$$A(\alpha) = \frac{\Gamma(\alpha-1)}{4\pi^2} \cos\left(\frac{\alpha\pi}{2}\right), \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function, κ is the spatial wave number, α is the power law exponent, \tilde{C}_n^2 is the generalized structure parameter with $m^{3-\alpha}$ as its unit, The function $A(\alpha)$ maintains the consistency between the index structure function and its power spectrum.

Consider a power spectrum that accounts for the inner scale and the outer scale of the anisotropic atmospheric turbulence. It can be defined as follows [6]:

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$$\Phi_n(\kappa, \alpha, \zeta) = A(\alpha) \widetilde{C}_n^2 \zeta^2 [\zeta^2 \kappa_{xy}^2 + \kappa_z^2 + \kappa_0^2]^{-\frac{\alpha}{2}} \times \exp\left(-\frac{\zeta^2 \kappa_{xy}^2 + \kappa_z^2}{\kappa_m^2}\right), \quad (3 < \alpha < 4), \quad (3)$$

where ζ is the anisotropic factor; $\kappa_0 = \frac{2\pi}{L_0}$ and L_0 is the outer scale parameter; $\kappa_m = \frac{C(\alpha)}{l_0}$ and l_0 is the inner scale parameter; $\kappa = \sqrt{\zeta^2(\kappa_x^2 + \kappa_y^2) + \kappa_z^2}$ and $\kappa_x, \kappa_y,$ and κ_z are the components of κ in $x, y,$ and z direction; $C(\alpha) = [\Gamma(\frac{5-\alpha}{2})A(\alpha)\frac{2}{3}\pi]^{\frac{1}{\alpha-5}}$.

Assuming a horizontal propagation path, one can ignore κ_z and set $\kappa_z = 0$. In this case, Eq. (3) can be defined as the one in [3], which is formed by multiplying the generalized von Karman model with the anisotropic factor $\zeta^{2-\alpha}$. The resulting expression is then the modified anisotropic non-Kolmogorov power spectrum and it is defined as follows [15]:

$$\Phi_n(\kappa, \alpha, \zeta) = A(\alpha) \widetilde{C}_n^2 \zeta^{2-\alpha} [\kappa^2 + \tilde{\kappa}_0^2]^{-\frac{\alpha}{2}} \times \exp\left(-\frac{\kappa^2}{\tilde{\kappa}_m^2}\right), \quad (\kappa > 0, 3 < \alpha < 4), \quad (4)$$

where $\tilde{\kappa}_0^2 = \kappa_0^2/\zeta^2$ and $\tilde{\kappa}_m^2 = \kappa_m^2/\zeta^2$.

3. NEW EXPRESSIONS FOR WAVE STRUCTURE FUNCTIONS

The WSFs for the isotropic non-Kolmogorov turbulence can be expressed as follows [1]:

$$D_{pl}(\rho, \alpha) = 8\pi^2 k^2 L \int_0^\infty \kappa \Phi_n(\kappa, \alpha) [1 - J_0(\kappa\rho)] d\kappa, \quad (5)$$

where $D_{pl}(\rho, \alpha)$ is the plane wave structure function; ρ is the scalar separation distance between two points in 2-D plane; $k = 2\pi/\lambda$ is the optical wave number; $\xi = 1 - z/L$ and L is the path length; z is the propagation distance; and $J_0(\cdot)$ is the zero order Bessel function of the first kind.

In this paper, Eq.(4) is used as the expression for the anisotropic non-Kolmogorov power spectrum in our derivation of new wave structure function expressions for optical waves. Then, Eq.(5) can be rewritten as follows:

$$D_{pl}(\rho, \alpha, \zeta) = 8\pi^2 k^2 L \int_0^\infty \kappa \Phi_n(\kappa, \alpha, \zeta) [1 - J_0(\kappa\rho)] d\kappa, \quad (6)$$

By substituting Eq.(4) into Eq.(6) and expanding $J_0(\cdot)$ as a series representation, one can obtain the new WSF expression for the plane waves as follows:

$$D_{pl}(\rho, \alpha, \zeta) = 8\pi^2 k^2 L A(\alpha) \widetilde{C}_n^2 \zeta^{2-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!(1)_n} \cdot \left(\frac{\rho}{2}\right)^{2n} \times \int_0^\infty \kappa^{2n+1} [\kappa^2 + \tilde{\kappa}_0^2]^{-\frac{\alpha}{2}} \cdot \exp\left(-\frac{\kappa^2}{\tilde{\kappa}_m^2}\right) d\kappa \quad (7)$$

Here, the integration can be resolved by using the confluent hypergeometric function of the second kind defined as follows [16], and the new expression of the wave structure function for the plane waves is defined as follows:

$$D_{pl}(\rho, \alpha, \zeta) = 4\pi^2 k^2 L A(\alpha) \widetilde{C}_n^2 \zeta^{2-\alpha} \times \left\{ \tilde{\kappa}_m^{2-\alpha} \Gamma\left(1 - \frac{\alpha}{2}\right) [1 - J(\alpha)] - \frac{\rho^2 \tilde{\kappa}_0^{4-\alpha}}{(\alpha-2)(4-\alpha)} \right\}, \quad (8)$$

$$J(\alpha) = {}_1F_1\left(1 - \frac{\alpha}{2}; 1; -\frac{\rho^2 \tilde{\kappa}_m^2}{4}\right), \quad (9)$$

where ${}_1F_1(\cdot)$ is the confluent hypergeometric function of the first kind in [16].

4. NEW EXPRESSIONS FOR SPATIAL COHERENCE RADIUS

In this section, the new expressions for SCRs of the plane waves and the spherical waves are derived. The SCRs derivations begin with the new WSFs derived in the previous section. Those WSFs need to be approximated and simplified to be used effectively in the numerical computations when one performs computer simulations.

Consider the WSF in Eq.(8). This equation involves the confluent hypergeometric functions ${}_1F_1(\cdot)$. The hypergeometric function can be approximately expressed as follows [16]:

$${}_1F_1(a; c; -z) \approx \begin{cases} 1 - \frac{az}{c}, & |z| \ll 1, \\ \frac{\Gamma(c)}{\Gamma(c-a)} z^{-a}, & \text{Re}(z) \gg 1, \end{cases} \quad (10)$$

Eq.(10) can be substituted into Eq.(8) and the WSFs of the optical waves can be rewritten as follows:

$$D_{\text{pl}}(\rho, \alpha, \zeta) \approx \begin{cases} D_{\text{pl}}(\rho, \alpha, \zeta) R(\alpha) \tilde{\sigma}_R^2(\alpha) [\tilde{\kappa}_m^{2-\alpha} \Gamma(1 - \frac{\alpha}{2}) - \frac{2^{2-\alpha} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \rho^{\alpha-2} - \frac{\rho^2 \tilde{\kappa}_0^{4-\alpha}}{(\alpha-2)(4-\alpha)}], \rho \gg l_0, \\ R(\alpha) \tilde{\sigma}_R^2(\alpha) \rho^2 [\frac{2-\alpha}{8} \tilde{\kappa}_m^{4-\alpha} \Gamma(1 - \frac{\alpha}{2}) - \frac{\tilde{\kappa}_0^{4-\alpha}}{(\alpha-2)(4-\alpha)}], \rho \ll l_0, \end{cases} \quad (11)$$

where $R(\alpha) = -0.5\alpha(\sin \frac{\pi\alpha}{4})^{-1} \zeta^{2-\alpha} k^{\frac{\alpha}{2}-1} L^{1-\frac{\alpha}{2}} [\Gamma(1 - \frac{\alpha}{2})]^{-1}$, and $\tilde{\sigma}_R^2(\alpha)$ is the non-Kolmogorov Rytov variance defined by the plane waves scintillation index in non-Kolmogorov turbulence [17] as follows:

$$\tilde{\sigma}_R^2(\alpha) = -8\pi^2 A(\alpha) \frac{1}{\alpha} \Gamma(1 - \frac{\alpha}{2}) \tilde{C}_n^2 k^{3-\frac{\alpha}{2}} L^{\frac{\alpha}{2}} \sin \frac{\pi\alpha}{4}, \quad (12)$$

From the WSF of the optical waves, the SCR ρ_0 is defined by the 1/e point of the complex degree of coherence [1] and $D(\rho_0, L) = 2$.

Based on the approximation expression of the WSF for the plane waves defined in Eq.(11), the new SCR expression of plane waves for the case of $L_0 = \infty$ is derived and it is defined as follows:

$$\rho_0 \equiv \rho_{\text{pl}} \approx \begin{cases} \left\{ \frac{2^{\alpha-2} \Gamma(\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} [\tilde{\kappa}_m^{2-\alpha} \Gamma(1 - \frac{\alpha}{2}) - \frac{2}{R(\alpha) \tilde{\sigma}_R^2(\alpha)}] \right\}^{\frac{1}{\alpha-2}}, l_0 \ll \rho_{\text{pl}} \ll L_0, \\ \left\{ R(\alpha) \tilde{\sigma}_R^2(\alpha) \frac{2-\alpha}{16} \tilde{\kappa}_m^{4-\alpha} \Gamma(1 - \frac{\alpha}{2}) \right\}^{\frac{1}{2}}, \rho_{\text{pl}} \ll l_0 \ll L_0, \end{cases} \quad (13)$$

5. EVALUATIONS USING NUMERICAL ANALYSIS

There are two sets of new expressions derived and they are set for evaluation with respect to various characterizing parameters. The WSF is defined in Eq.(10) and the SCR is defined in Eq.(13). We have made some general assumptions in the numerical simulations: the optical waves propagate with the generalized structure parameter $\tilde{C}_n^2 = 1.4 \times 10^{-14} \text{m}^{3-\alpha}$; the wavelength $\lambda = 1.65 \times 10^{-6} \text{m}$; the scalar separation distance is $\rho = 3 \text{cm}$, the inner scale of the eddy size is 1 mm, the outer scale of the eddy size is 10 m; the optical path lengths vary from 100 m to 8 km; and the power law exponent α varies from 3 to 4; and the case of $l_0 \ll \rho \ll L_0$ is used for the SCR simulations.

5.1. Evaluations on WSFs

The first set of simulations are performed using the new expressions of wave structure function defined in Eq.(10). The focus of the evaluation is to analyze the behaviors of the WSFs in terms of various characterization parameters. Those include the power law exponent α , the turbulence strength $\tilde{\sigma}_R^2$, and the anisotropic factor ζ .

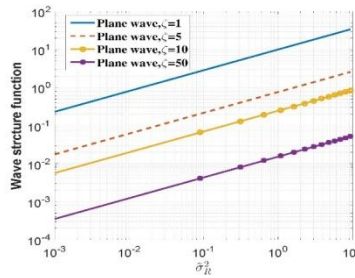


Figure. 1. Wave structure function of plane wave with respect to turbulence strength anisotropic factor.

Figure 1 shows the behavior of the WSF with respect to the increasing turbulence strength $\tilde{\sigma}_R^2$ and the increasing anisotropic factor ζ . For the plane wave, the WSFs increase as the turbulence strength gets stronger. Also it is also important to note that the WSFs are significantly influenced by the anisotropic factor ζ that the strength of the WSFs gets as much as 400 times weaker as the anisotropy increases from 1 to 50.

5.2. Evaluation on SCRs

The second set of simulations is performed using the new expressions of spatial coherence radiuses defined in Eq.(13). Similar to the simulations using the WSFs, the focus of the evaluation is also to analyze the behaviors of SCRs in terms of various characterization parameters. Those include the power law exponent α , the turbulence strength $\tilde{\sigma}_R^2$, and the anisotropic factor ζ .

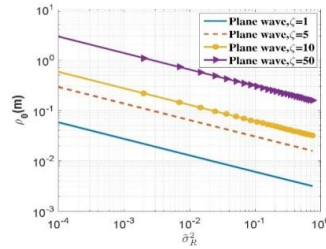


Figure 2. Coherence radius of plane wave as a function of turbulence strength and anisotropic factor.

Figure 2 shows the behavior of the SCR of the plane with respect to the increasing turbulence strength $\tilde{\sigma}_R^2$ and the increasing anisotropic factor ζ . On the contrary to the WSFs cases, the SCRs decrease as the turbulence strength gets stronger. Also it is important to note that the SCRs are also significantly influenced by the anisotropic factor ζ that the strength of the SCRs get as much as 40 times stronger as the anisotropy increases from 1 to 50.

6. CONCLUSION

In this work, we have presented new sets expressions for the wave structure functions and also for the spatial coherence radiuses of the free space optical waves such as the plane waves propagating in a horizontal path of a free space, which is disturbed by an anisotropic turbulence. Those newly derived analytic expressions of WSFs and SCRs are evaluated and their behaviors are observed by varying three major charactering parameters, which are the power law exponent α , the turbulence strength $\tilde{\sigma}_R^2$, and the anisotropic factor ζ .

Those three parameters individually or in their combinations have extensive impacts on the magnitudes of the WSFs and SCRs. The behaviors of the WSFs and the SCRs come out differently with respect to the power law exponent. Also with respect to the increasing turbulence strength, the WSFs and SCRs show an inverse relation that the WSFs increase while the SCRs decrease. Similarly, the anisotropic factor affects the WSFs and the SCRs inversely.

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